MATH 2028 Curl and Divergence

GOAL: Introduce three classical differential operators (grad, curl, div) and study their relationship.

Recall: Given a function $f: \mathcal{U} \to \mathbb{R}$ defined on an open set $\mathcal{U} \subseteq \mathbb{R}^n$, the gradient of f is a vector field (defined on the same $\mathcal{U}$), denoted by ∇f (or $\text{grad}(f)$):

$$\nabla : \{\text{functions}\} \xrightarrow{\text{grad}} \{\text{vector fields}\}$$

where $\nabla f = \left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \ldots, \frac{\partial f}{\partial x_n} \right)$

Next, we define another differential operator which takes a vector field to a function.

Given a vector field $\mathbf{F}: \mathcal{U} \to \mathbb{R}^n$ defined on an open set $\mathcal{U} \subseteq \mathbb{R}^n$, the divergence of $\mathbf{F}$ is a function (defined also on $\mathcal{U}$), denoted by $\text{div}\,\mathbf{F}$ (or $\nabla \cdot \mathbf{F}$)

$$\text{div} : \{\text{vector fields}\} \xrightarrow{\text{div}} \{\text{functions}\}$$

where $$\text{div}\, F = \frac{\partial F_1}{\partial x_1} + \frac{\partial F_2}{\partial x_2} + \dots + \frac{\partial F_n}{\partial x_n}$$

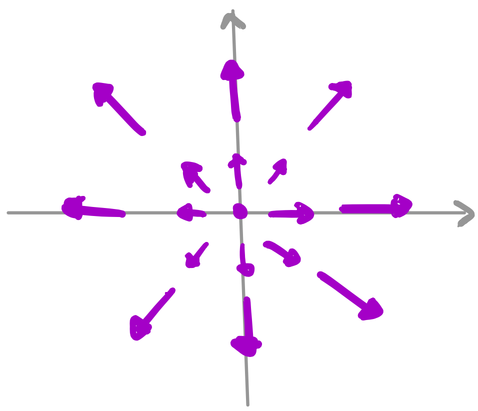
Here, $F = (F_1, F_2, \dots, F_n)$ in components.

Example:

(1) $F = (x_1, \dots, x_n)$ radial vector field

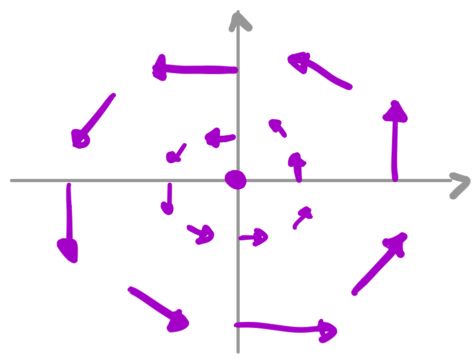
$$\nabla \cdot F = \frac{\partial x_1}{\partial x_1} + \frac{\partial x_2}{\partial x_2} + \dots + \frac{\partial x_n}{\partial x_n}$$

$$\equiv n$$



(2) $n=2$: $F(x,y) = (-y, x)$ rotational vector field

$$\nabla \cdot F = \frac{\partial(-y)}{\partial x} + \frac{\partial x}{\partial y} \equiv 0$$



Prop: $\nabla \cdot (\nabla f) = \Delta f$

Pf: $$\nabla \cdot (\nabla f) = \nabla \cdot \left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \dots, \frac{\partial f}{\partial x_n}\right)$$

$$= \frac{\partial^2 f}{\partial x_1^2} + \frac{\partial^2 f}{\partial x_2^2} + \dots + \frac{\partial^2 f}{\partial x_n^2} =: \Delta f$$

$\blacksquare$

Finally, we define the curl of a vector field in dimension three (we will generalize this to higher dimensions later):

$$\text{curl} : \begin{Bmatrix} \text{vector} \\ \text{fields} \end{Bmatrix} \xrightarrow{\text{curl}} \begin{Bmatrix} \text{vector} \\ \text{fields} \end{Bmatrix}$$

where
$$\text{curl}\, F = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$$

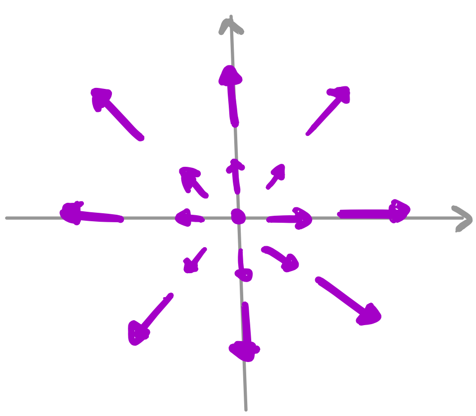
(Notation: $\text{curl}\, F = \nabla \times F$)

$$= \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z},\ \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x},\ \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right)$$

Example: $(n=3)$

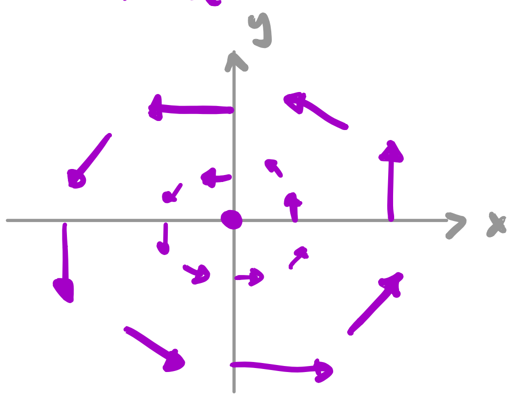
(1) $F = (x, y, z)$ radial vector field

$$\nabla \times F = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix}$$



$$= (0, 0, 0)$$

(2) $F(x,y,z) = (-y, x, 0)$ rotational vector field



$$\nabla \times F = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -y & x & 0 \end{vmatrix}$$

$$= (0, 0, 2)$$

$\square$

Prop: In $\mathbb{R}^3$, we have

(i) $\nabla \times (\nabla f) \equiv \vec{0}$ $\forall$ function f

(ii) $\nabla \cdot (\nabla \times F) \equiv 0$ $\forall$ vector field F

Proof: Exercise!